Quantum Cosmology with a Complex ϕ^4 Field at Finite Temperature

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Quantum cosmology with a complex ϕ^4 field at finite temperature for the Vilenkin boundary condition and the Hartle–Hawking boundary condition is studied. The Euclidean region in minisuperspace is generally bounded by a closed curve. The wave function of the universe calculated in the WKB approximation can cross, through scattering or tunneling, from the initial classically allowed region to the final classically allowed region. For a given value of scalar field, the probability density for the Vilenkin boundary condition is smallest at zero temperature, and when temperature increases, the probability density rises remarkably. At the same time, the opposite results are given for Hartle–Hawking boundary condition. In addition, the classical trajectories of the universe are calculated in detail.

1. INTRODUCTION

Since Hartle and Hawking (1983) and Vilenkin (Vilenkin, 1982, 1988) described the birth of the universe in the frame of quantum gravitation, quantum cosmology has become an area of extensive research. Many significant results have been obtained through the discussion of the universe wave function (Halliwell, 1988; Ducan, 1990). Generally speaking, the ultimate aim of quantum cosmology is to predict the initial state of the universe by the wave function of the universe. The universe wave function is formulated by solving the zero-energy Schrödinger equation, i.e., the Wheeler–DeWitt (WDW) equation (DeWitt, 1967). Two methods, which correspond to different boundary conditions, have been applied to the problem of determining the quantum state of the universe. The first approach, suggested by Vilenkin

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(1982, 1983, 1984, 1986, 1988), is based on the fact that, after the universe spontaneously nucleates in a de Sitter space, it evolves along an inflationary scenario. The universe was created through a potential barrier, which is generally referred to as "quantum tunneling from nothing." The universe wave function, i.e., the tunneling wave function, derived in this way is complex. Another approach is given by Hartle, Hawking, *et al.* (Hartle and Hawking, 1983; Hawking, 1982; Hawking and Luttrell, 1984a, b; Hawking and Wu, 1985; Hawking and Page, 1986; Halliwell and Hawking, 1985; Halliwell, 1986, 1987). The wave function of the universe, i.e., the Hartle–Hawking wave function, is given by a path integral over compact Euclidean geometry and is real. Vilenkin (1988) pointed out that the tunneling wave function can predict initial states that lead to inflation. More detailed descriptions have been given for some cosmological models with matter fields (Hartle and Hawking, 1983; Vilenkin, 1982, 1983, 1984, 1986, 1988, 1994, 1995; Ducan, 1990; Hawking 1982; Hawking and Luttrell, 1984a, b; Hawking and Wu, 1985; Hawking and Page, 1986; Halliwell and Hawking, 1985; Halliwell, 1986, 1987, 1988; Kamenshchik *et al.*, 1995). Generally, the complex scalar model has more physical significance than the real scalar model because it coincides with the matter hydrodynamic field model. So Khalatnikov and Mezhlumian (1992) introduced a complex scalar field in the frame of Friedmann cosmology. Different from the real scalar field case, the classically forbidden region in the minisuperspace is bounded by a closed curve and this region is not convex everywhere. Amendola and Khalatnikov (1994) further studied the complex field. Using the Vilenkin boundary condition and WKB approximation, they obtained a wave function of the universe which can, by scattering or tunneling, reach the final classical state from the initial classical state. For cosmological constant $\Lambda = 0$, the probability density distribution describes a classical universe that preferentially starts with a large matter content. In addition, they checked the no-rolling approximation method through a slow-rolling approximation. The result shows that the no-rolling approximation is reliable for $\left| U' \right| \ll U$. But all these models for the scalar field are considered at zero temperature. According to the big-bang cosmological model, the universe in the very early period is in a state with high temperature and high density, and spontaneous symmetry breaking will be restored when $T > T_c$. Therefore, it is very important to explore quantum cosmology at finite temperature and to understand the very early state of the universe. So we further explore the effect of quantum cosmology with a complex ϕ^4 field at finite temperature in this paper.

In Section 2 we describe the behavior of the complex ϕ^4 field at finite temperature, and give the classical trajectories of the universe in Section 3. The wave function of the universe and probability density distribution are calculated in Section 4. Finally, we present our conclusions in Section 5.

2. COMPLEX ϕ^4 FIELD AT FINITE TEMPERATURE

The effective potential of the ϕ^4 field at zero temperature takes the form

$$V(\phi) = -m^2\phi^2/2 + \phi^4/4\phi_0 + \phi_0 m^4/4 \tag{1}$$

where ϕ_0 is the value which satisfies the equation $V(\phi) = 0$. The effective potential of the ϕ^4 field at finite temperature can be found by using the Linde procedure (Linde, 1979), and reads

$$V(\phi, T) = M\phi^2/2 + \phi^4/4\phi_0 + \phi_0 m^4/4 \tag{2}$$

where $M = T^2/4\phi_0^2 - m^2$. Setting $\partial V(\phi, T)/\partial\phi = 0$, we have

$$M\phi + \phi^3/\phi_0 = 0 \tag{3}$$

which has three roots

$$\phi_1 = 0, \qquad \phi_{2,3}^2 = \phi_0(m^2 - T^2/4\phi_0^2) \tag{4}$$

where ϕ_1 corresponds to a false vacuum and ϕ_2 and ϕ_3 correspond to two true vacua. If $m^2 - T^2/4\phi_0^2 < 0$, ϕ_2 and ϕ_3 reduce to two imaginary roots and the true vacua will vanish. So the condition

$$m^2 - T^2/4\phi_0^2 = 0 \tag{5}$$

determines the critical temperature T_c. When $T > T_c$, the spontaneous symmetry breaking is restored. From Eq. (5), we obtain $T_c = 2m\phi_0$.

3. THE CLASSICAL TRAJECTORIES OF EVOLUTION OF THE UNIVERSE

We shall consider the Lagrangian action of an interacting gravitational and complex scalar field ($\dots$, 19)

$$S = \int (R/16\pi G + g^{\mu\nu}\,\Phi^*_{;\mu}\,\Phi_{;\nu}/2 - V(\Phi^*\Phi))\,\sqrt{-g}\,d^4x \tag{6}$$

where g is the metric determinant and R is the scalar curvature. Similar to Amendola and Khalatnikov (1994), we assume that the effective potential $V(\Phi^*\Phi)$ includes a cosmological constant Λ and the signature of the metric is $(+ ---)$. The complex scalar field is $\Phi = \phi\exp(i\theta)$, and the effective potential $V(\Phi^*\Phi)$ takes the form of a ϕ^4 field at finite temperature

$$V(\Phi^*\Phi) = V(\phi, T) = M\phi^2/2 + \phi^4/4\phi_0 + \phi_0 m^4/4 + \Lambda \tag{7}$$

where $M = T^2/4\phi_0^2 - m^2$. The metric and complex scalar are assumed to be homogeneous and isotropic. For a closed Robertson–Walker universe, the metric is written as

$$ds^2 = dt^2 - a^2(t)\, d\Omega_3^2 \tag{8}$$

where $d\Omega_3^2$ is the metric on a unit three-sphere and $a(t)$ is the scale factor. So the scalar curvature is given by

$$R = 6a^{-2}(1 + \dot{a}^2 + a\ddot{a}) \tag{9}$$

Considering (8) and (9), Eq. (6) reduces to

$$S = \int L(a, \dot{a}, \phi, \dot{\phi}, \theta, \dot{\theta})\, dt \tag{10}$$

where

$$L = -12\pi^2 l_p^{-2} a\dot{a}^2 + 12\pi^2 l_p^{-2}\, a + \pi^2 a^3 \dot{\phi}^2 + \pi^2 a^3 \phi^2 \dot{\theta}^2 - 2\pi^2 a^3 V(\phi, T)$$

Using the method of canonical quantization, we can get the momenta conjugate to the minisuperspace coordinates a, ϕ, and θ, respectively:

$$p_a = -24\pi^2 l_p^{-2} a\dot{a}, \qquad p_\phi = 2\pi^2 a^3 \dot{\phi}, \qquad p_\theta = 2\pi^2 a^3 \phi^2 \dot{\theta} \tag{11}$$

The current of the scalar field is defined by

$$J^\mu = g^{\mu\nu}\phi^2 \theta_{;\nu} \tag{12}$$

So $J^0 = \phi^2 \dot{\theta}$, and the classical conserved charge is

$$Q = \int J^0 \sqrt{-g}\, d^4 x = 2\pi^2 a^3 \phi^2 \dot{\theta} \tag{13}$$

Using (10), (11), and (13), we can derive the Hamiltonian of the system

$$H = -p_a^2/48\pi^2 l_p^{-2} a + p_\phi^2/4\pi^2 a^3 + Q^2/4\pi^2 a^3 \phi^2 - 12\pi^2 l_p^{-2} a \tag{14}$$
$$+ 2\pi^2 a^3 V(\phi, T)$$

The operator transformation is given by

$$p_a \rightarrow -i\partial/\partial a, \qquad p_\phi \rightarrow -i\partial/\partial\phi \tag{15}$$

So the Wheeler–DeWitt equation, i,e., $H\Psi = 0$, satisfies

$$[\nabla_a^2 - 12 l_p^{-2} a^{-2}\, \nabla_\phi^2 - U(a, \phi, T)]\Psi = 0 \tag{16}$$

where

$$U(a, \phi, T) = a^2[-12 l_p^{-2} Q^2/a^4 \phi^2 + 576\pi^4 l_p^{-4} - 96\pi^4 l_p^{-2} a^2 V(\phi, T)]$$

In order to compare with the literature (Amendola and Khalatnikov, 1994), we use units in which $8\pi G = \hbar = c = 1$ and rescale every term of the

superpotential $U(a, \phi, T)$. Thus the Wheeler–DeWitt equation can now be rewritten as

$$[\nabla_a^2 - 6a^{-2}\,\nabla_\phi^2 - U(a, \phi, T)]\Psi = 0 \qquad (17)$$

where $U(a, \phi, T) = a^2[-Q^2/a^4\phi^2 + 1 - a^2V(\phi, T)]$. The first and third terms of $U(a, \phi, T)$ coincide roughly with the initial classically allowed region and final one, respectively, and the middle term coincides with the classically forbidden region.

Figure 1 gives the contours of the equipotential in the minisuperspace (a, ϕ) for superpotential $U(a, \phi, T)$. Figure 1a shows the results at different temperatures in the case of $Q = 5$, and the Fig. 1b in the the case of $Q = 0$. In the minisuperspace (a, ϕ), $U(a, \phi, T) = 0$ divides the plane into two regions: $U(a, \phi, T) > 0$, corresponding to the classically forbidden region, i.e., Euclidean region; and $U(a, \phi, T) < 0$, corresponding to the classically allowed region, i.e., the Lorentzian region. $U(a, \phi, T) = 0$ determines the boundary between these two regions. We can see from this figure that, as in the case of the effective potential $V(\phi, T) = m^2\phi^2/2 + \Lambda$ introduced in the literature (Amendola and Khalatnikov, 1994), the Euclidean region contour at finite temperature is also a closed curve in minisuperspace (a, ϕ). With the increase of temperature the Euclidean region contracts gradually. In addition, some classical trajectories cannot enter the Euclidean region. When they approach the boundary $U(a, \phi, T) = 0$, they bounce back to become the collapsed universe. There are, however, some classical trajectories which need not cross the classically forbidden region and always keep expanding. When $Q = 0$, the superpotential of the complex scalar field becomes identical to that for a real scalar field in which the classically forbidden region is an open curve. The lines from top to bottom in Fig. 1b correspond to the case of temperature $T = 0$, $T = m\phi_0$, $T = 2m\phi_0$, $T = 3m\phi_0$, and $T = 3.42m\phi_0$, respectively.

Using the equation of motion, $\dot{p}_\phi = -\partial H/\partial \phi$, and the conserved equation of current, $J^\mu_{;\mu} = 0$, we can write the classical field equation as

$$\ddot{\phi} + 3\dot{a}\dot{\phi}/a - \phi\dot{\theta}^2 + V'(\phi, T) = 0, \qquad (\sqrt{-g}\phi^2\theta^{;\mu})_{;\mu} = 0 \quad (18)$$

where a prime denotes derivative with respect to ϕ. So the classical trajectory in the initial classically allowed region is

$$a(\phi) = a_1 \exp[(\phi^2 - \phi_1^2)/4] \qquad (19)$$

where a_1 and ϕ_1 are initial values in this region. Equation (19) is identical to the result given by Amendola and Khalatnikov (1994). Similarly, the classical trajectory in the final classically allowed region is
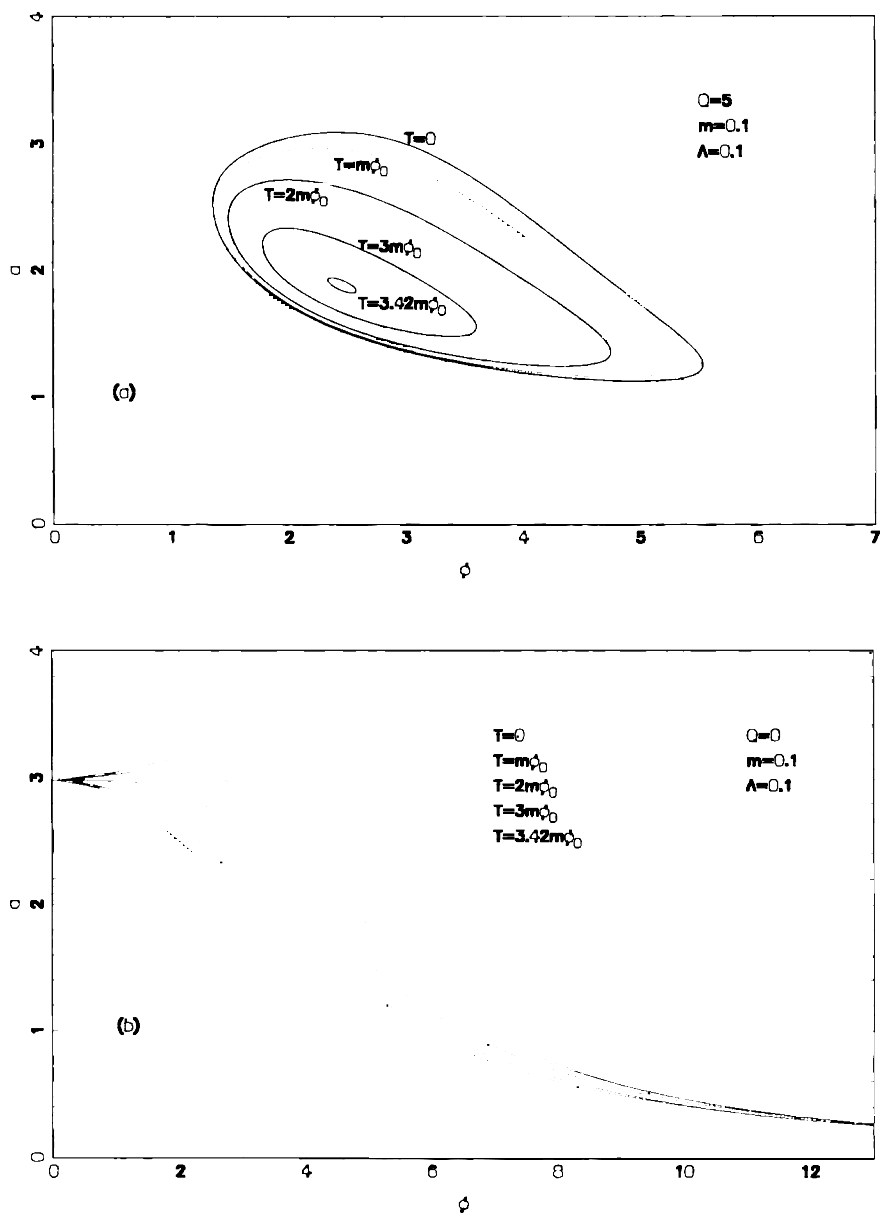
Fig. 1. Contours of the equipotential in the minisuperspace (a, ϕ) for superpotential $U(a, \phi, T)$. (a) The case $Q = 5$; (b) the case $Q = 0$.

$$a(\phi) = a_2[I(\phi_2) - I(\phi)] \exp[(\phi_2^2 - \phi^2)/8] \qquad (20)$$

where

$$I(\phi) = \frac{-4(\Lambda + \phi_0 m^4/4)\ln \phi}{0.04 - T^2\phi_0^2}$$

$$+ \frac{(-0.0016\phi_0 + 0.08T^2\phi_0^3 - T^4\phi_0^5 + 64\Lambda) \ln(4x^2 - 0.04\phi_0 + T^2\phi_0^3)}{32(0.04 - T^2\phi_0^2)}$$

Here a_2 and ϕ_2 are initial values in the region.

4. THE WAVE FUNCTION OF THE UNIVERSE AND PROBABILITY DENSITY DISTRIBUTION

Adopting the no-rolling approximation, we can approximate the Wheeler–DeWitt equation (17) as

$$[\nabla_a^2 + (p/a)\, \nabla_a - U(a, \phi, T)]\Psi = 0 \qquad (21)$$

where p denotes the ambiguity in the ordering of factors a and $\partial/\partial a$. Taking into account the WKB approximation, we obtain the solution of Eq. (21) at $U(a, \phi, T) < 0$,

$$\Psi_V = C_1 a^{-p/2} p^{-1/2}(a, \phi, T) \exp\left[\pm i \int p(a, \phi, T)\, da \right] \qquad (22)$$

The solution of Eq. (21) in the range $U(a, \phi, T) > 0$ is

$$\Psi_V = C_2 a^{-p/2} \big|p(a, \phi, T)\big|^{-1/2} \exp\left[\pm \int \big|p(a, \phi, T)\big|\, da \right] \qquad (23)$$

In the two expressions above, c_1 and c_2 are coefficients and $p(a, \phi, T) = [-U(a, \phi, T)]^{1/2}$. The negative and positive signs in Eq. (22) correspond to *outgoing* and *ingoing* waves, i.e., expanding and contracting universes, respectively. The exact solutions of Eqs. (22) and (23) are not easily obtained, so we divide the minisuperspace into different regions to solve them. As mentioned before, the first and third terms of the superpotential $U(a, \phi, T)$ coincide roughly with the initial and final classical allowed regions, respectively, and the middle term coincides with classical forbidden one. The value of a at the boundary between the regions with the first and second terms is determined by $a_{12}^2 \cong Q/\phi$, while the value of a at the boundary between the regions with the second and third terms is $a_{23}^2 \cong 1/V(\phi, T)$. Because the superpotential $U(a, \phi, T)$ at finite temperature is a closed curve in the minisuperspace (a, ϕ), we can find two points which satisfy $a_{12}^2 \cong$

a_{23}^2 with roots ϕ_l and ϕ_h. When $\phi > \phi_h$ and $0 < \phi < \phi_l$, the Euclidean region will vanish, which means that the wave function never goes through a potential barrier. In these regions, the middle term of $U(a, \phi, T)$ is negligible. At $a_* = (Q/\phi)^{1/3} V(\phi, T)^{-1/6}$, the remaining two terms of $U(a, \phi, T)$ become equal. From Eqs. (22) and (23), it is clear that the wave function in the classically forbidden region is exponential and that in the classically allowed region is oscillating.

Inserting superpotential $U(a, \phi, T)$ in Eqs. (22) and (23), we can calculate the wave function. In the range $a < a_{12}$, the wave function is

$$\Psi_V = a^{-p/2}(a\phi/Q)^{1/2} \exp[\pm i(Q/\phi) \ln(a/a_{12})] \tag{24}$$

which is an exponential wave function. The wave function in the range $a > a_{12}$ follows,

$$\Psi_V = \frac{a^{(-p-1)/2}}{[1 - V(\phi, T)a^2]^{1/4}} \times \exp\left[\pm \frac{(1 - V(\phi, T)a^2)^{3/2} - (1 - V(\phi, T)a_{12}^2)^{3/2}}{3V(\phi, T)} \right] \tag{25}$$

So we can also evaluate the wave function in the range $a > a_{23}$

$$\Psi_V = \frac{-ia^{(-p-1)/2}}{[V(\phi, T)a^2 - 1]^{1/4}} \exp\left[\pm\left(\frac{-(1 - V(\phi, T)a_{12}^2)^{3/2}}{3V(\phi, T)} \right) \right] \times \exp\left[\pm\frac{-i(V(\phi, T)a^2 - 1)^{3/2}}{3V(\phi, T)} \right] \tag{26}$$

In the regions $\phi > \phi_h$ and $0 < \phi < \phi_l$, the wave functions read

$$\Psi_V = a^{-p/2}(a\phi/Q)^{1/2} \exp[\pm i(Q/\phi) \ln(a/a_*)], \qquad a < a_* \tag{27}$$

$$\Psi_V = a^{(-p-2)/2} V(\phi, T)^{-1/4} \exp[\pm i V(\phi, T)^{1/2} (a^3 - a_*^3)/3], \qquad a > a_* \tag{28}$$

The connection formulas for wave functions in different regions are the same as those of Amendola and Khatnikov (1994). According to Halliwell (1988) and Vilenkin (1994), the relation of transformation from the tunneling wave function to the Hartle–Hawking wave function is

$$\Psi_{HH} = \Psi_V(V \to e^{-i\pi}V, a \to e^{i\pi/2}a) \tag{29}$$

From Eqs. (26) and (29), we can also calculate the tunneling wave function in the range $a > a_{23}$,

$$\Psi_{HH} = \frac{-ia^{(-p-1)/2}e^{i\pi(p-1)/4}}{[V(\phi, T)a^2 - 1]^{1/4}} \exp\left[\pm\left(\frac{(1 - V(\phi, T)a_{12}^2)^{3/2}}{3V(\phi, T)}\right)\right]$$

$$\times \exp\left[\pm\frac{i(V(\phi, T)a^2 - 1)^{3/2}}{3V(\phi, T)}\right] \qquad (30)$$

The probability density distribution for ϕ at a given value of a takes the form

$$P(\phi) = \frac{i}{2} a^p[\Psi^*\nabla_a\Psi - \Psi\nabla_a\Psi^*] \qquad (31)$$

Obviously, for regions $0 < \phi < \phi_l$ and $\phi > \phi_h$, the probability density distribution for the Vilenkin boundary condition is equal to 1. Substituting Eq. (26) into Eq. (31), we obtain the probability density distribution for the Vilenkin boundary condition for the remaining intermediate-ϕ region where the WF tunnels the potential barrier,

$$P_V(\phi) = \exp[-2(1 - V(\phi, T)Q/\phi)^{3/2}(3V(\phi, T))^{-1}] \qquad (32)$$

Similarly, inserting Eq. (30) into Eq. (31), we obtain the probability density distribution for the Hartle–Hawking boundary condition in the same ϕ region,

$$P_{HH}(\phi) = \exp[2(1 - V(\phi, T)Q/\phi)^{3/2}(3V(\phi, T))^{-1}] \qquad (33)$$

The probability density distribution for the Vilenkin boundary condition at different temperatures is shown in Fig. 2a. The variation of the probability density with scalar field ϕ is analogous to that of Amendola and Khalatnikov (1994). Given an arbitrary value of the scalar field ϕ in the range (ϕ_l, ϕ_h), the probability density at zero temperature is smallest. With the increase of temperature T, the probability density grows remarkably. This result is also easily obtained by theoretical analysis. Figure 2b gives the probability density distribution for the Hartle–Hawking boundary condition at different temperatures, which is opposite to that for the Vilenkin boundary condition.

5. CONCLUSIONS

In this paper we have explored the classical and quantum cosmology of the complex ϕ^4 field at finite temperature; our conclusions are summarized as follows: (1) The Euclidean region in minisuperspace (a, ϕ) is generally a closed curve. (2) The wave function of the universe calculated in the WKB approximation can cross, through scattering or tunneling, from the initial classically allowed region to the final classically allowed region. (3) The probability density for the Vilenkin boundary condition at zero temperature
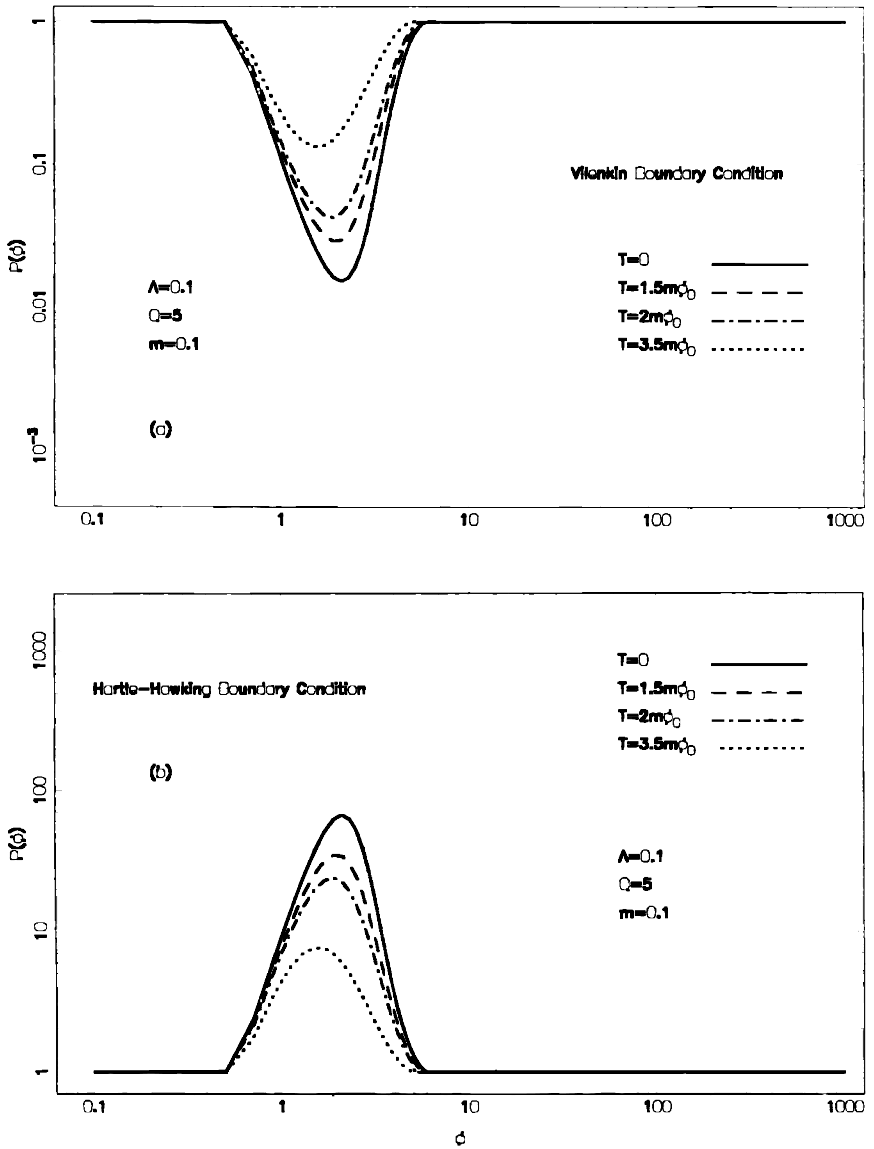
Fig. 2. The probability density distribution at different temperatures (a) for the Vilenkin boundary condition and (b) for the Hartle–Hawking boundary condition.

is smallest at every given value of the scalar field. As temperature T grows, the probability density increases remarkablly. Compared with this result, the probability density distribution for the Hartle–Hawking boundary condition shows contrary behavior.

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REFERENCES

Amendola, L., Khalatnikov, *I. M.*, *et al.*, *Phys. Rev. D* **49** (1994), 1881.
Dewitt, B. S., *Phys. Rev.* **160** (1967), 1113.
Ducan, M. T., Quantum geometrodynamics, Preprint (1990), UMN-TH-916/90.
Halliwell, J. J., *Nucl. Phys. B* **266** (1986), 228.
Halliwell, J. J., *Nucl. Phys. B* **286** (1987), 729.
Halliwell, J. J., Quantum cosmology: An introductory review, Preprint (1988), NSF-ITP-88-131.
Halliwell, J. J., and Hawking, S. W., *Phys. Rev. D* **31** (1985), 1777.
Hartle, J. B., and Hawking, S. W., *Phys. Rev. D* **28** (1983), 2960.
Hawking, S. W., *Pontif. Accad. Sci. Scr. Varia* **48** (1982), 563.
Hawking, S. W., *Nucl. Phys. B* **239** (1984), 257.
Hawking, S. W., *Nucl. Phys. B* **239** (1986), 257.
Hawking, S. W., and Luttrell, J. C., *Phys. Lett. B* **143** (1984a), 83.
Hawking, S. W., and Luttrell, J. C., *Nucl. Phys. B* **247** (1984b), 250.
Hawking, S. W., and Page, D. N., *Nucl. Phys. B* **264** (1986), 185.
Hawking, S. W., and Wu, Z. C., *Phys. Lett. B* **151** (1985), 15.
Kamenshchik, A. Yu., Khalatnikov, I. M., *et al.*, *Phys. Lett. B* **357** (1995), 36.
Khalatnikov, I. M., and Mezhlumian, A., *Phys. Lett. A* **169** (1992), 308.
Linde, A. D., *Rep. Prog. Phys.* **42** (1979), 389.
Vilenkin, A., *Phys. Lett. B* **117** (1982), 25.
Vilenkin, A., *Phys. Rev. D* **27** (1983), 2848.
Vilenkin, A., *Phys. Rev. D* **30** (1984), 509.
Vilenkin, A., *Phys. Rev. D* **33** (1986), 3560.
Vilenkin, A., *Phys. Rev. D* **37** (1988), 888.
Vilenkin, A., *Phys. Rev. D* **50** (1994), 2581.
Vilenkin, A., *Phys. Rev. Lett.* **74** (1995), 846.